CONVEXITY AND A CERTAIN PROPERTY P_m

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ABSTRACT

The property P_m (directly analogous to Valentine's property P_3) is used to prove several curious results concerning subsets of a topological linear space, among them the following: (a) If a closed set S has property P_m and contains k points of local nonconvexity no distinct pair of which can see each other via S, then S is the union of $m - k - 1$ or fewer starshaped sets. (b) Any closed connected set with property P_m is polygonally connected. (c) A closed connected set S with property P_m is an L_{m-1} set (each pair of points may be joined by a polygonal arc of $m - 1$ of fewer sides in S). (d) A finite-dimensional set with property P_m is an L_{2m-3} set. A new proof of Tietze's theorem on locally convex sets is given, and various examples refute certain plausible conjectures.

1. Introduction. A considerable amount of research has been devoted to the convexity properties of a set which are determined by assumptions made on each m points of the set ($m \geqq 2$). By altering those assumptions for a closed set in E^2, for example, one may conclude variously that S is starshaped (Krasnosel'skiĭ [10]), S is the union of three convex sets (Valentine [16]), S is convex (Marr and Stamey [11]), S is the union of two convex sets (Stamey and Marr [14]), or S is the union of two starshaped sets (Koch and Marr [9]). For related results see Allen [1], Hare and Gaddum [6], Hare and Kenelly [7], and McKinney [13].

Our concern here is with the following condition: In any linear space, a set S is said to be (m, n) *convex* if it contains at least m points, and if for each m distinct points of S at least n of the $\binom{m}{2}$ possible segments determined by those points are contained in S. If S is $(m,1)$ convex, or briefly, *m-convex*, then S is said to have *property* P_m. We shall call a set *exactly* (m,n) *convex* if it is (m,n) convex, but not $(m,n+1)$ convex, $m \geqq 3$ and $n \leqq \binom{m}{2}$, and a set is *exactly m-convex* if it is m-convex, but not $(m-1)$-convex, $m \geqq 2$. The convention that no nonempty set

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is 1-convex will be made, so it follows that *every infinite convex set is exactly 2-convex.*

The concept of (m,n) convexity is but a special case of even more general ideas proposed by J. E. Allen [1] and various concepts discussed by Danzer, Grünbaum, and Klee [4], but these authors were primarily interested in characterizations of convexity. Property P_m was introduced for the case $m = 3$ by Valentine [16] who proved it was a sufficient condition for a closed set in E^2 to be the union of three convex sets. It is apparently a difficult question whether an arbitrary closed set in E^d having property P_m for $m > 3$ and $d \geqq 2$ is the union of even a finite number of convex sets.

The discussion will begin with the more general concept of (m,n) convexity, then it will be devoted to property P_m. The setting is a Hausdorff topological linear space $\mathscr{L}$ over the reals, with certain results applying more particularly to a finite d-dimensional space E^d. The (closed) segment with endpoints x and y is the set $\{z_\lambda = (1-\lambda)x + \lambda y\colon 0 \leqq \lambda \leqq 1\}$, denoted by xy, while the half-open segments determined by x and y are the sets $\{z_\lambda\}$ for which, respectively, $0 \leqq \lambda < 1$, $0 < \lambda \leqq 1$, and $0 < \lambda < 1$, denoted by $[xy),(xy]$, and (xy). The convex hull of a set S will be denoted $\operatorname{conv} S$, its (topological) closure, $\operatorname{cl} S$, and its boundary, $\operatorname{bd} S$.

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2. (m,n) Convexity. For brevity let the binomial coefficient $\binom{m}{2} = \frac{1}{2}m(m-1)$ be represented by C_m. For each pair of integers (m,n), $m \geqq 3$ and $1 \leqq n < C_m$ one can easily construct sets $S \subset E^2$ which are exactly (m,n) convex, in the following manner: Note that the positive integers $1,2,3,\cdots$ may be written $C_3 - 2$, $C_3 - 1$, $C_4 - 3$, $C_4 - 2$, $C_4 - 1$, $C_5 - 4,\cdots$ and thus there exists integers r and s such that $n = C_{m-r} - s$, $0 \leqq r \leqq m - 3$ and $1 \leqq s \leqq m - r - 1$; then take S to be the set consisting of r isolated points, together with an $(m-r)$-sided convex polygon with interior, but with precisely s open sides removed. However, an elementary argument we leave for the reader proves the following result, which shows the impossibility of such a onstruction for *closed* sets.

LEMMA 1. *A closed $(m, C_{m-1} + 1)$ convex set is convex.*

It is possible to sharpen this considerably, which yields the following theorem:

THEOREM 1. *If $S \subset \mathscr{L}$ and S is a closed (m,n) convex set with $n > \frac{1}{4}(m-1)^2$,* *then S is either convex or the union or a closed, convex set S_1 and k isolated points not in S_1, where*

$$k \leqq m - \tfrac{1}{2}(1 + \sqrt{8\,n+1.}$$

PROOF. If S is not convex there exist points $x \in S$ and $y \in S$ such that $xy \not\subset S$ and if S is connected we can find two infinite nets $\{x_i\}$ and $\{y_j\}$ in S defined by the directed sets D and E such that $\lim_{i\in D} x_i = x$ and $\lim_{j\in E} y_j = y$. Since S is closed there exist $i_0 \in D$, $j_0 \in E$ such that for $i \geqq i_0$ and $j \geqq j_0$ $x_i y_j \not\subset S$. We can then obtain m points x_i and y_j in S such that the number of segments joining them and lying in S does not exceed

$$2C_{m/2} = \tfrac{1}{4}(m-1)^2 - \tfrac{1}{4}$$

if m is even, or

$$C_{r+1} + C_r = \tfrac{1}{4}(m-1)^2$$

if m is odd and $m = 2r+1$, which contradicts the hypothesis. If S is not connected, then by the preceding argument each component is convex. In view of the inequality $n > \frac{1}{4}(m-1)^2$ it follows that the components must be singletons $\{p_1\}, \cdots, \{p_k\}$ for some finite k, and S_1, any closed convex set. Since among the points p_i no joins occur while among any set of points in S_1 all possible occur, we must have

$$C_{m-k} \geqq n$$

from which the desired inequality in m, k, and n may be derived.

COROLLARY 1. *A closed, connected (m,n) convex set in $\mathscr{L}$ is convex if $n > \frac{1}{4}(m-1)^2$.*

REMARK 1. The result in Corollary 1 is best possible since the union of any two infinite convex sets is $(m, [\frac{1}{4}(m-1)^2])$ convex for each $m \geqq 2$. Obviously, the bound for k in the theorem is also best by the manner in which it was obtained. A bound which depends on m only is useful if S is exactly (m, n) convex. Since $n > \frac{1}{4}(m-1)$ it follows that

$$k < m - \tfrac{1}{2}(1 + \sqrt{2m^2 - 4m + 3}).$$

REMARK 2. The referee has pointed out that a theorem due to Turán [15] enables one to determine the (m,n) convexity of a closed set $S \subset \mathscr{L}$ in terms of the k-convexity of its components and the number of its one-pointed components

(= isolated points). If S has infinitely many connected components, or if a many-pointed component of S is not k-convex for any k, then S is not m-convex nor (m, n) convex for any m, n. Suppose S has s isolated points and a positive number t of many-pointed components $C_1, \cdots, C_t$, and that C_i is exactly $(m_i + 1)$-convex for $1 \leqq i \leqq t$ $(0 \leqq s < \infty, 1 \leqq m_i < \infty$ for $1 \leqq i \leqq t)$. Then S is exactly $(M + s + 1)$-convex, where $M = \sum_{i=1}^{t} m_i$. Suppose $M = s < m < \infty$, and let $m - s = kM + r$, where $k \geqq 1$ and $1 \leqq r \leqq M$. Then S is exactly (m, n) convex, where $n = \frac{1}{2}(m - s)(k - 1) + \frac{1}{2}r(k + 1)$. That S is not $(m, n + 1)$ convex can be shown explicitly by a simple construction: First choose M points $y_1, \cdots, y_M$ in $\bigcup_{i=1}^{t} C_i$, such that $y_i y_j \not\subset S$ for $1 \leqq i < j \leqq M$, then replace each y_i by $k + 1$ (if $1 \leqq i \leqq r$) or k (if $r < i \leqq M$) distinct points sufficiently close to y_i, as in the proof of Theorem 1, and finally, add all the isolated points of S. The (m, n) convexity of S follows from Turán's theorem: A graph with $m - s$ vertices and fewer than n edges must contain an independent set of more than M vertices.

3. Starshaped sets, L_n sets and m-convexity. We now turn our attention to the more special case of m-convexity. There are sets which are not m-convex for any $m \geqq 2$; for example, consider the complement of a strictly convex body in E^d, $d > 1$. On the other hand, the union of any collection of m many-pointed convex sets is $(m + 1)$-convex $(m \geqq 1)$. The converse need not hold as the example of the five-pointed star with interior shows. It is an open question whether for any value of m greater than 3 a closed m-convex set in E^2 is the union of a finite number of convex sets.

The following usual terminology will be employed freely:

DEFINITION: If the open segment joining two points x and y is contained in S, then we shall say that x *sees* y *via* S (x and y need not be in S, and we use the convention that any point in S sees itself via S). If x sees $y \in T$ via S then we shall say that x *sees* T *via* S. A set T is *visually independent via* S if no two distinct members of T see each other via S. If $x \in S$, define the *local kernel of* S *at* x to be the set of all points of S which x can see via S, and denote this by S_x. S is said to be *locally starshaped at* $x \in S$ if there exists a neighborhood whose intersection with S is starshaped with respect to x; S is *locally starshaped* if and only if it is locally starshaped at each of its points (it is well known that an open set in a topological linear space is locally starshaped). Finally, we define a *point of local nonconvexity* (or, *lnc point*) of S to be any point $x \in S$ such that each neighborhoood U of x contains points $y \in S$, $z \in S$ such that $yz \not\subset S$. Such points are referred to in Valentine [17] as points of "strong local nonconvexity" (Definition 4.2, p. 48).

In connection with our opening comments in this section, the following example proves to be quite instructive:

EXAMPLE 1. Let E^2 be identified with the complex plane and let C be the unit circle $|z| = 1$, with $z_n = e^{\pi i/2^n}$ for $n = 0, 1, 2, \cdots$. Let P be the infinite-sided polygon which circumscribes C, touching C at precisely the points 1, $e^{-\pi i/2}$, and z_n for n even. The set S is then defined as the set of points on and inside P with those z_n deleted for which n is odd.

We show that S is 4-convex, but is not the union of finitely many convex sets (an example of a 4-convex starshaped set in E^3 which is not the union of finitely many convex sets can be obtained by taking the cone of p over S, where p is any point not in the plane of S; see the definition preceding Theorem 7). Note that S is the disjoint union of C less the points z_{2n+1} $(n = 0, 1, \cdots)$, the interior of C, and connected sets T_{2n+1} $(n = -2, -1, 0, \cdots)$ which contain the vertices of P, with $z_{2n+1} \in \operatorname{bd} T_{2n+1}$ $(n = 0, 1, \cdots)$. It then follows that if $x \in T_p$ and $y \in T_q$, $p \neq q$, the segment xy is either tangent to C or cuts C in two points $x' \in \operatorname{bd} T_p$ and $y' \in \operatorname{bd} T_q$. Thus if $z_r \in xy$ and r is odd, then $r = p$, or $r = q$. Now suppose x_i $(i = 1, 2, 3, 4)$ are four points of S such that $x_i x_j \not\subset S$ $(1 \leqq i < j \leqq 4)$. Then at least three of those points belong to $\bigcup_{n=0}^{\infty} T_{2n+1}$, say x_1, x_2, x_3, and no segment $x_i x_j$ is tangent to C. Suppose $x_1 \in T_p$, $x_2 \in T_q$, and $x_3 \in T_r$, and that $z_u \in x_1 x_2$, $z_{u'} \in x_1 x_3$ and $z_{u''} \in x_2 x_3$. Since the points x_1, x_2, x_3 cannot be collinear, u, u', and u'' are distinct and we must have $u = p$ or $u = q$, $u' = p$ or $u' = r$, and $u'' = q$ or $u'' = r$. The cases being similar, assume that $u = p$ and, therefore, $u' \neq p$, $u' = r$, and $u'' = q$. A contradiction is thereby gained by examining the cases resulting from $z_v \in x_1 x_4$, $z_{v'} \in x_2 x_4$, and $z_{v''} \in x_3 x_4$. Thus S is 4-convex. If S were the finite union of convex sets, say $S = \bigcup_{k=1}^{m} C_k$, then the ray with origin 0 passing through z_p $(p = 1, 3, 5, \cdots)$ meets bd S again at w_p, and a subsequence $w_{p_1}, \cdots, w_{p_n}, \cdots$ must belong to a single C_k. Since $\lim_{n \to \infty} w_{p_n} = w_\infty = 1$, the triangle $(w_{p_1}, w_{p_2}, w_\infty)$ contains z_{p_2} in its interior; thus, there exists an s such that $z_{p_2} \in \operatorname{conv}\{w_{p_2}‘ w_{p_2}, w_{p_s}\} \subset S$, a contradiction.

The following results show that m-convex sets may be represented as finite unions of starshaped sets, however. If S is exactly m-convex, S contains a maximal visually independent subset $X = \{x_1, \cdots, x_{m-1}\}$. Since each point of $S \setminus X$ must see some x_i and x_i can see itself via S,

$$S = \bigcup_{i=1}^{m-1} S_{x_i}.$$

Hence:

THEOREM 2. *Any m-convex set is the union of $m-1$ or fewer starshaped sets.*

REMARK 3. The starshaped subsets S_{x_i} of Theorem 2 need not be m-convex, as the set in Example 1 shows: Take $x_1 = 2z_1 - z_3$, and S_{x_1} is clearly not m-convex *for any finite m.* The result in Theorem 2 is clearly best possible since S can be the union of $m-1$ disjoint closed convex sets. It is best possible even for connected m-convex subsets of E^2, as shown by the following elementary example (to be used later for another purpose):

EXAMPLE 2. With the usual coordinatization of E^2, take the points $a_k = (k,0)$ for $k = 1,\cdots,m$ and $b_k = (k,1)$ for $k = 1,\cdots,m-1$. Define $T_k = \operatorname{conv}\{a_k, b_k, a_{k+1}\} \setminus (a_k a_{k+1}) \setminus (b_k a_{k+1})$ and $S = \cup_{k=1}^{m-1} T_k \setminus \{a_m\}$. S is connected and m-convex, but is not the union of fewer than $m-1$ starshaped sets since no two of the $m-1$ points $b_1,\cdots,b_{m-1}$ can belong to the same local kernel of S.

Theorem 2 can be improved for closed, connected m-convex sets as the next result shows.

THEOREM 3. *If a closed m-convex set $S \subset \mathscr{L}$ contains k lnc points $(k \geqq 0)$ which are visually independent via S, then S is the union of $m-k-1$ or fewer starshaped sets*(1).

PROOF. If $k = 0$, Theorem 2 implies the result directly, so assume $k \geqq 1$ and let $q_1,\cdots,q_k$ be k visually independent lnc points of S. Choose a maximal visually independent subset $X = \{x_1,\cdots,x_h\}$ of S such that $x_i = q_i$ for $i = 1,\cdots,k (k \leqq h)$. As before,

$$S = \bigcup_{i=1}^{h} S_{x_i}.$$

Now since q_1 is an lnc point of S there exist nets $\{y_i\}$ and $\{z_j\}$ in S over the directed sets D, E such that $\lim_{i\in D} y_i = \lim_{j\in E} z_j = q_1$ but $y_i z_j \not\subset S$ for all $i \in D$, $j \in E$. Since S is closed there exist $i \in D$ and $j \in E$ such that $u_1 = y_i$ and $v_1 = z_j$ c annot

(1) It follows that in all cases $k \leqq m-1$, and by applying the argument in the proof of the theorem this may easily be improved to $k \leqq [\frac{1}{2}(m-1)]$. If in addition S is connected, Tietze's theorem (below) implies that $k \geqq 1$, if S is not convex.

see $x_1, \cdots, x_h$ via S. Thus $\{u_1, v_1, x_2, x_3, \cdots, x_h\}$ is a visually independent set via S. Proceeding inductively, we may locate pairs $(u_2, v_2), \cdots, (u_k, v_k)$ corresponding to $q_2, \cdots, q_k$ such that

$$\{u_1, v_1, u_2, v_2, \cdots, u_k, v_k, x_{k+1}, \cdots, x_h\}$$

is a visually independent subset of S. By m-convexity, $h + k \leqq m - 1$, and the theorem follows.

(Simple examples may be constructed to show that the number $m - k - 1$ in Theorem 3 is best possible.)

Several authors have explored the convexity properties of L_n sets—sets having the property that each pair of points may be joined by a polygonal arc in the set having n or fewer sides (see [2], [8], [12] and [18]). This concept has an intimate relationship with m-convexity as we shall see.

LEMMA 2. *A closed m-convex set S is locally starshaped. Thus a closed connected m-convex set is polygonally connected.*

PROOF. If there exists a point x and a net $\{x_i\}$ in S over D such that $\lim_{i \in D} x_i = x$ but $x_i x \not\subset S$ for all $i \in D$ then there is an $i_1 \in D$ such that for $i \geqq i_1$ $x_1 x_i \not\subset S$. Set $y_1 = x_1$ and $y_2 = x_{i_1}$. In the same manner there is an $i_2 \in D$ such that $i_2 \geqq i_1$ and for $i \geqq i_2$, $y_2 x_i \not\subset S$. If we set $y_3 = x_{i_2}$ then $\{y_1, y_2, y_3\}$ is a visually independent subset of S. Continuing inductively, one can find an infinite visually independent subset $\{y_1, \cdots, y_m, \cdots\}$, contradicting the m-convexity of S. The second part of the lemma then follows by standard arguments (see for example Valentine [17], Theorem 4.3, p. 49).

(Note that Example 1 shows the necessity of the restriction to closed sets in Lemma 2.)

THEOREM 4. *Any closed, connected m-convex set $S \subset \mathscr{L}$ is an L_{m-1} set.*

PROOF. Let x and y be points of S. Since S is polygonally connected there is a polygonal arc $P \subset S$ joining x and y. Let $F = E^d$ be a finite-dimensional subspace containing P and suppose S' is the component of $S \cap F$ which contains P. Then S' is a closed m-convex subset of S lying in a finite-dimensional linear space E^d. If we prove there is a polygonal arc in S' joining x and y and having $m - 1$ or fewer sides, we shall be finished.

Since S' is closed there is a polygonal arc P' in S' joining x and y having s or fewer sides, where s is the number of sides in P, and having minimal length. Let the consecutive vertices of P' be written

$$x = x_0, x_1, \cdots, x_n = y, \qquad n \leqq s,$$

where the notation is chosen so that no three consecutive vertices are collinear. Consider a point $y_i \in (x_{i-1}x_i)$ for any i, $1 \leqq i < n$. Since $y_ix_{i+1} \not\subset S'$ (otherwise there would exist a polygonal arc shorter than P') and S' is closed there is a $y_{i+1} \in (x_ix_{i+1})$ depending on y_i such that $y_iy_{i+1} \not\subset S'$. Thus, points $y_i \in (x_{i-1}x_i)$, $i = 1, \cdots, n$, may be chosen inductively so that for each $i = 1, \cdots, n-1$, $y_iy_{i+1} \not\subset S'$. But also, because of the minimal length of P', $y_iy_j \not\subset S'$ for any $j > i+1$ and hence $\{y_1, \cdots, y_n\}$ is a visually independent subset of S'. By m-convexity, $n \leqq m-1$.

(A polygonal arc with $m-1$ sides shows that the result of Theorem 4 is best possible.)

Our methods provide an interesting proof of Tietze's theorem on local convexity. Define a set to be *locally convex* if it contains no points of local nonconvexity. This concept corresponds to "weak local convexity" in Valentine [17] (Definition 4.2).

THEOREM 5 (TIETZE): *If $S \subset \mathscr{L}$ and S is closed, connected, and locally convex, then S is convex.*

PROOF. The classic argument shows that S is polygonally connected. Choose any two points $x \in S$ and $y \in S$ and let P be a polygonal arc in S joining x and y. There is a finite-dimensional subspace F, and thus a compact convex set $N \subset F$, which contains P. Let S' be the component of $N \cap S$ which contains P. Then S' is a compact, connected, locally convex subset of F, and accordingly, one may cover S' by relatively open convex neighborhoods $N_x \subset S'$ $(x \in S')$. Let $N_{x_1}, \cdots, N_{x_{m-1}}$ be a finite subcover of the covering $\{N_x\}$. Hence, as the union of the $m-1$ convex sets $N_{x_1}, \cdots, N_{x_{m-1}}$, S' is m-convex. Among all polygonal arcs in S' joining x and y and having m or fewer sides, let P' have least length. Then, as in the proof of Theorem 4, P' has $m-1$ or fewer sides. It follows that if P' is not a segment it has at least three consecutive noncollinear vertices, x_1, x_2, and x_3, with x_2 an lnc point of S', since otherwise there would exist a polygonal arc P'' with m or fewer sides and of length less than that of P'. Hence $xy = P' \subset S' \subset S$, proving that S is convex.

Polygonal connectedness for connected m-convex sets which are not closed may also be derived, as well as a result analogous to Theorem 4, provided the finite-dimensionality of the space be required. The first step is to prove a result which replaces the first part of Lemma 2.

LEMMA 3. *If x is a limit point of an m-convex set $S \subset E^d$, then x can see $S \setminus \{x\}$ via S.*

PROOF. We use induction on d. Since x is a limit point of S there is an infinite sequence $X_0 = (x_i)$ in S, $i = 1,2,\cdots$, converging to x. The m-convexity of S implies there is an x_{i_1} which can see via S all members of a subsequence $X_1 \subset X_0$, an $x_{i_2} \in X_1$ which can see via S all members of a subsequence $X_2 \subset X_1$, and inductively, there is an $x_{i_n} \in X_{n-1}$ which can see via S all members of a subsequence $X_n \subset X_{n-1}$, $n = 1,2,\cdots$. If $d = 1$, it immediately follows that $(xx_{i_1}] \subset S$ (a special conclusion for dimension 1 not deducible in higher dimensions). If $d > 1$, let L be a flat of dimension $d-2$ containing x. It may be assumed that no subsequence of X_0 lies in a flat of dimension less than d or else the induction hypotheseis implies the result (by intersecting the flat with S, forming an m-convex set of lower dimension). Applying this to the particular subsequence (x_{i_k}) defined above, there exist three points $u = x_{i_r}$, $v = x_{i_s}$, and $w = x_{i_t}$ $(r < s < t)$ such that the hyperplanes $H(u)$, $H(v)$, and $H(w)$ uniquely determined by L and the respective points u, v, w are pairwise distinct. Hence one pair among u, v, and w, say u and v, is strictly separated by the hyperplane determined by the remaining point. Since at most finitely many members of X_t lie in $H(w)$, there is a subsequence X'_t of X_t which lies entirely on one side of $H(w)$, and hence is strictly separated by $H(w)$ from one of the points u or v, say u. Then ux_j meets $H(w)$ at a point $y_j \in S$ for each $x_j \in X'_t$, and $\lim_j y_j = x$. The induction hypothesis applied to the set $S \cap H(w) = S'$ then implies that x can see $S' \setminus \{x\}$ via S', completing the proof.

REMARK 4. If S be the set of all terminating sequences of the form $(x_1,\cdots,x_n,0,0,\cdots)$ in Hilbert space $\mathscr{H}$, then S is a convex subset of $\mathscr{H}$ whose closure is $\mathscr{H}$, and no point in $\mathscr{H} \setminus S$ can see S via S. Lemma 3 cannot hold, therefore, for infinite-dimensional spaces.

THEOREM 6. *In a finite-dimensional linear space every connected m-convex set is polygonally connected.*

PROOF. Let P be a maximal polygonally-connected subset of S. Unless $P = S$, it follows that both P and $S \setminus P$ are $(m-1)$-convex sets. Since either $\mathrm{cl}\,P \cap (S \setminus P)$ or $P \cap \mathrm{cl}(S \setminus P)$ must be nonvoid, either a point of $S \setminus P$ can see P via P or a point of P can see $S \setminus P$ via $S \setminus P$, contradicting the maximality of P. Hence $P = S$ and the theorem follows.

COROLLARY 2. In a finite-dimensional linear space every connected m-convex set is an L_{2m-3} set.

PROOF. Let x and y be two points of the given set S, and let

$$x = x_0, x_1, \cdots, x_n = y$$

be the consecutive vertices of a polygonal arc in S joining x and y such that the number n of sides is minimal among all such arcs joining x and y. If $n \geqq 2m - 2$ then $\{x_{2i}\colon\ i = 0, 1, \cdots, m-1\}$ would be a visually independent subset of S, a violation of m-convexity. Therefore, $n \leqq 2m - 3$.

The set S of Example 2 shows that the preceding result is the best possible. For, S is m-convex, but every polygonal arc in S joining a_1 with b_{m-1} has at least $2m - 3$ sides.

REMARK 5. The referee has pointed out that certain results of this section hold with only minor alterations for the following weaker condition: An infinite set is *∞-convex* if it contains no infinite visually independent subset; a set is *exactly ∞-convex* if it is ∞-convex but not m-convex for finite m.

The m-convexity of a set obviously implies it is ∞-convex, but the example illustrated by Figure 1 represents a compact starshaped subset of E^2 which is exactly ∞-convex. The set S_{x_1} mentioned previously (the local kernel at $x_1 = 2z_1 - z_3$ of the 4-convex set S defined in Example 1) is not even ∞-convex. The conclusion of Theorem 2 holds for ∞-convex sets if "$m - 1$ or fewer" is replaced by "finitely many", the proof itself requiring no change at all. Lemmas 2 and 3 and Theorem 6 each hold without change for ∞-convex sets (the conclusions regarding L_n sets reduce to mere polygonal connectedness in the case of ∞-convex sets, so Theorem 4 and Corollary 2 do not lead to any new results).

The next theorem is a generalization of the well-known proposition that the cone of a point over any convex set is convex (we shall use this classical result in the proof). Here, the setting is a vector space $\mathscr{V}$ over any ordered field.

DEFINITION If $S \subset \mathscr{V}$, the *cone of v over S* is defined to be the set $\bigcup_{x \in S} vx$, and will be denoted vS. If $V \subset \mathscr{V}$, the *cone of V over S* is the set $\bigcup_{v \in V} vS$, denoted VS.

Recall that the *kernel* K of a set S is the intersection of all the local kernels of $S (= \bigcap_{x \in S} S_x)$. It is well known that K is convex and may be obtained as the intersection of all the maximal convex subsets of S.

THEOREM 7. *If $S \subset \mathscr{V}$ is m-convex and $V \subset \mathscr{V}$ is any set such that the segment joining each pair of distinct points of V meets the kernel K of S, then the set*

$S' = VS$ is also m-convex. Further, if S is the union of n convex sets, then S' is the union of n convex sets.

PROOF. Let K' be the kernel of S'; we prove that $V \subset K'$. If $v_1 \in V$ and $x \in S'$, there are points $v_2 \in V$ and $y \in S$ such that $x \in v_2 y$. If $v_1 = v_2$ then $v_1 x \subset v_1 y \subset S'$. Otherwise, there exists a point $k \in v_1 v_2 \cap K$ and hence $py \subset S$. Now with the notation $u[vw]$ denoting the cone of u over vw,

$$\begin{aligned} v_1 x \subset \operatorname{conv}\{v_1, v_2, y\} &= y[v_1 v_2] = y[v_1 k \cup k v_2] \\ &= y[v_1 k] \cup y[k v_2] = v_1[yk] \cup v_2[yk] \subset S'. \end{aligned}$$

Thus v_1 can see any point of S' via S' and hence belongs to K'.

Now suppose $x_1, \cdots, x_m$ are any m points of S'. By definition, there exist points $v_i \in V$ and $y_i \in S$ such that $x_i \in v_i y_i$, $i = 1, \cdots, m$. By m-convexity of S there exist i, j, $i \neq j$, such that $y_i y_j \subset S$. If $\Delta_1 = v_i[y_i y_j]$ then Δ_1 is a convex subset of S', and the set $\Delta_2 = v_j \Delta_1$ is also convex. Since $v_j \in K'$, $\Delta_2 \subset S'$. But then $x_i \in \Delta_1 \subset \Delta_2$ and $x_j \in \Delta_2$, so $x_i x_j \subset \Delta_2 \subset S'$, proving that S' is m-convex.

The remainder of the theorem is obviously an application of the result just obtained (for the special case of 2-convexity) to each of the convex sets $C_i' = KC_i$, where the C_i are the convex sets in the union $S = \bigcup_{i=1}^{n} C_i$. Thus, VC_i' is convex for each i and hence

$$S' = VS = V\,[\bigcup_{i=1}^{n} C_i] = V\,[\bigcup_{i=1}^{n} KC_i] = \bigcup_{i=1}^{n} VC_i'.$$

4. The concept of m-convexity and finite unions of convex sets. McKinney [13] and Stamey and Marr [14] have given characterizations of closed sets which are unions of two convex sets. It would seem that the concept of m-convexity should be a useful tool in a characterization of sets which are unions of finitely many convex sets. Example 1 shows that if one attempts to use m-convexity as the only criterion the restriction to closed sets is necessary. Valentine's result concerning P_3 [16] suggests the conjecture that a closed m-convex set is the union of m or fewer convex sets. The following example shows that this is false for $m > 3$, even in E^2:

EXAMPLE 3. Take the set in E^2 as illustrated in Fig. 2, consisting of the union of 6 parallelograms and their interiors. This set is closed and 4-convex, yet is not a union of fewer than 5 convex sets. The m-sided ring-shaped analogue of Fig. 2 (as illustrated in Fig. 3 for the case $m = 6$) is $(m+1)$-convex, but is not a union of fewer than $[\frac{1}{2}(3m+1)]$ convex sets. A stronger example however, is provided by k disjoint copies of Figure 2, which can be altered slightly to achieve connectedness; such a set is closed and $(3k+1)$-convex, yet is not a union of fewer than $5k$ convex sets.

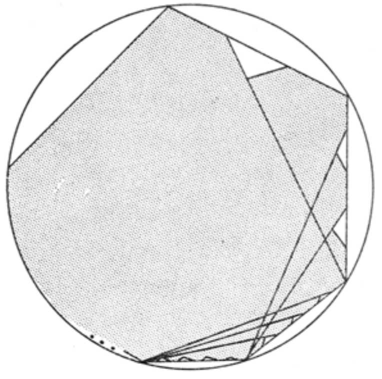

Fig. 1

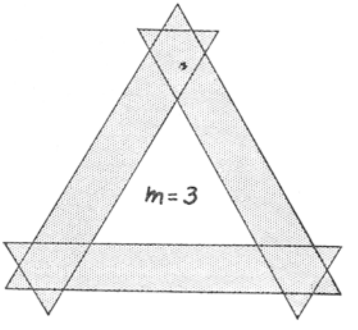

Fig. 2

Since a closed, connected 3-convex set is starshaped, one might consider imposing the condition that an m-convex set be starshaped. The next example shows that a closed, starshaped m-convex set need not be the union of m convex sets for $m > 4$ (it is an open question for the case $m = 4$).

EXAMPLE 4. The 10-pointed star with interior as illustrated in Fig. 4 is a closed, starshaped 5-convex set which is not the union of fewer than six convex sets. Note that this example consists of a superposition of two pentagonal stars and interiors. By taking the union of suitably positioned elongated pentagonal stars one can generalize the example to higher values of m.

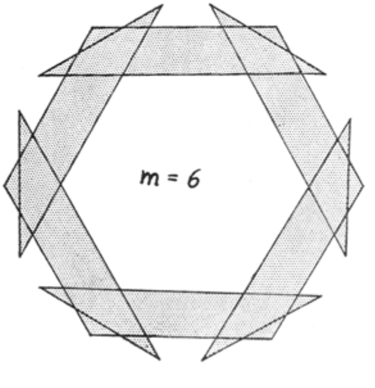

Fig. 3

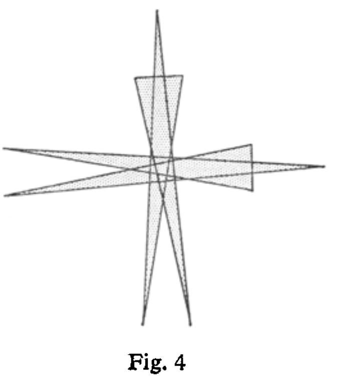

Fig. 4

A more concise example which achieves the same purpose is the following: Given $m \geqq 2$, take $n = 3m - 1$, $\varepsilon = e^{2\pi i/n}$, and $S_m = \bigcup_{j=1}^{n} \operatorname{conv}\{0, \varepsilon^j, \varepsilon^{j+m}\}$. S_m is an n-pointed star, is $(m+1)$-convex, and is not a union of fewer than $[3m/2]$ convex sets.

Restricted versions of Valentine's result have been obtained either in higher dimensions or with a larger value of m. E. Buchman [3] has proved that a compact 3-convex set $S \subset E^d$ $(d \geqq 3)$ whose set Q of lnc points is contained in the interior of the convex hull of S and whose kernel has nonempty interior is the union of *two* convex sets. Guay [5] proved that if S is a closed, starshaped 4-convex subset of E^2 whose kernel is one-dimensional then S is the union of 4 convex sets, and that if S is a closed, connected 4-convex subset of E^2 whose complement contains a bounded component, then S is the union of 5 convex sets (Example 3 shows that this result is best possible).

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